Forward-Backward Differential Equations: Approximation of Small Solutions

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Abstract

In the context of physics, economic dynamics, finance, optimal control, biology and other applied sciences, many mathematical models contain mixed type functional differential equations (MTFDEs), equations with both delayed and advanced arguments. Knowing from the analysis of delay differential equations (DDEs) that the evaluation of small solutions (that decay faster than any exponential) often leads to computational problems (degeneracy), we investigate this subject in the case of MTFDEs. Some computations have been carried out and are presented here, concerning the linear nonautonomous case. We continue this work and extend the investigation to other problems.

Key words: Mixed-type functional differential equation, method of steps, small solutions, collocation, least squares, finite element method.
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1 Introduction

In the context of physics, economic dynamics, finance, optimal control, biology and other applied sciences, many mathematical models contain forward-backward differential equations, usually denominated mixed type functional differential equations (MTFDEs):

$$x'(t) = F(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_n)), \tag{1}$$

where the shifts τ_i may take negative or positive values. The equation (1) is a multi-delay-advance differential equation form.

For the problems studied in this work, we focus our attention on computation of solutions. We are particularly interested in obtaining the numerical solution of the equations:

$$x'(t) = F(t, x(t), x(t-\tau), x(t+\tau)), \quad \tau > 0 \tag{2}$$

Interest in this type of equation is motivated by applications in optimal control [20] [21], economic dynamics [22], nerve conduction [6] and travelling waves in a spatial lattice [1], [19].

Important contributions in such fields have appeared in the literature in the two last decades of past century and it remains an actual subject of research until now.

Concerning the application of MTFDEs to optimal control and economic dynamics, a fundamental contribuition was introduced by Rustichini in 1989 [21]. However, the basis for the application of MTFDEs to optimal control was given earlier by Mischenko and his co-authors [20].

Nowadays, MTFDEs continue to play an important role in analysis and modeling of economic growth. The have been used, for example, by Boucekkine et al. in [5] to study the relation between demographic and economic variables in the vintage capital problem.

In 2009 [2] a system of functional differential equations having leads and lags has been applied by Albis et al. to model the dynamic behaviour of the capital growth in an overlapping-generations model with continous trading and finitely lived agents.

Differential equations with advanced and delayed time arguments may also arise in simple growth models with delays, such as models with investment generations lags proposed by Collard et al [7] in 2004. In this article, the authors proposed and implemented a numerical scheme which combine a Runge-Kutta and a shooting method to solve the short run dynamics of a neoclassical growth model with a simple time to build lag.

Another application of MTFDEs is investigated in [3, 4, 6, 16] where a nonlinear functional differential equation with delay and advanced arguments is used to model conduction in a myelinated nerve axon.

Special attention has been paid by many authors to mixed-type functional differential equations of the form

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-1) + \gamma(t)x(t+1), \tag{3}$$

where x is the unknown function, α, β and γ are known functions.

In 2005, Iakovleva and Vanegas [13] studied equation (3) in a particular case, presenting existence and uniqueness results. A similar approach has been followed by the authors of [8]. Inspired on their earlier work about delay differential equations [9, 17] and based on the existing insights on the qualitative behaviour of MTFDEs, they have developed a new approach to the analysis of these equations in the autonomous case. More precisely, they analysed MTFDEs as boundary value problems, that is, for a MTFDEs of the form (3), but with constant coefficients α, β and γ, they considered the problem of finding a differentiable solution on a certain real interval $[-1, k]$, $k \in \mathbb{N}$, given its values on the intervals $[-1, 0]$ and $(k-1, k]$. They concluded that in general the specification of such boundary functions is not sufficient to ensure that a solution can be found. With arbitrary boundary conditions, the problem turns out to be ill-posed. They provided sufficient analytical background on which the numerical scheme could be built. For the case where such a solution exists they introduced a numerical algorithm based on the θ-method to compute it. The same authors of [8] and their colaborators extended this work in [11]. In 2011, Da Silva and Escalante [23] presented a numerical algorithm based on the method of steps and tau method, for the approximation of solutions of BVP equation (3) in the autonomous case.

2 Numerical Experiments

We say that $x(t)$ is a small solution of equation (3) iff $t \to +\infty \lim x(t)e^{\lambda t} = 0$, $\forall \lambda \in \mathbb{R}^+$.

It is known that the detection and computation of small solutions are important problems when studying DDEs. Such problems have been discussed in [9, 10, 12, 17]. Therefore, we decided to investigate how our method for MTFDEs work in the case of a small solution.

We are concerned with the numerical solution of the linear homogeneous nonautonomous equation (3) in the case of a small solution.

The numerical algorithms applied to the solution of this problem, based on the collocation method and on the method of steps, were introduced in [24, 14, 25]. In [26, 15] new algorithms were introduced, based on the finite elements and on the least squares methods.

In the present work, the BVP under consideration is defined in Example 2.

The choosen boundary value problem (BVP) is defined by equations (4)-(6).

$$x'(t) = -2t\ x(t) - e^{-2t}x(t-1) + e^{2t}x(t+1), \tag{4}$$

with boundary conditions

$$x(t) = e^{-t^2}, \qquad t \in [-1, 0]; \tag{5}$$

$$x(t) = e^{-t^2}, \qquad t \in [k-1, k]. \tag{6}$$

The exact solution is $x(t) = e^{-t^2}$.

The numerical results of BVP (4)-(6) are presented in Table 1. There is a good agreement with the convergence order of the collocation method ($k = 2$). Similar results are obtained when the least squares and finite element methods are applied. We are still working with larger intervals of estimation ($k = 3, 4$).

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ϵ and estimates for the order p		
Step	$k=2$	
h_i	ϵ	p
$1/8$	$1.18101e-3$	
$1/16$	$2.94768e-4$	2.0024
$1/32$	$7.26304e-5$	2.0209
$1/64$	$1.79561e-5$	2.0161
$1/128$	$4.45928e-6$	2.0096

Table 1: Collocation method. Estimates of error ϵ and convergence order p for the approximate solution $\tilde{x}$ in the case $k=2$. $\epsilon = \|x-\tilde{x}\|_\infty$ (error on $[0,1]$).

References

[1] K. A. Abell, C. E. Elmer, A. R. Humphries, E. S. Van Vleck, *Computation of mixed type functional differential boundary value problems*, SIADS **4**, 3 (2005) , 755-781.

[2] H. D'Albis and E. Augeraud-Veron, *Competitive Growth in a Life-Cycle Model: existence and Dynamics*, International Review of Economics **50** (2009), 459–484.

[3] J. Bell, *Behaviour of some models of myelinated axons*, IMA Journal of Mathematics Applied in Medicine and Biology **1** (1984), 149-167.

[4] J. Bell and C. Cosner, *Threshold conditions for a diffusive model of a myelinated axon*, Journal of Mathematical Biology **18** (1983), 39-52.

[5] R. Boucekkine, D. de La Croix and O. Licandro, *Modeling Vintage Structures with DDEs: Principles and Applications*, Mathematical Population Studies **11** (2004), 151179.

[6] H. Chi, J. Bell and B. Hassard, *Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory*, Journal of Mathematical Biology **24** (1986), 583-601.

[7] F. Collard, O. Licandro, L.A. Puch, *The short-run dynamics of optimal growth models with delays*, Computing in economics and finance, 117, Society for Computational Economics (2004).

[8] N. J. Ford, P. M. Lumb, *Mixed-type functional differential equations: a numerical approach*, Journal of Computational and Applied Mathematics, **229** (2009),471-479.

[9] N. J. Ford, P. M. Lumb, *Numerical approaches to delay equations with small solutions*, E. A. Lipitakis(Ed), Proceedings of HERCMA 2001, 1, 101-108.

[10] N. J. Ford, S. M. Verduyn-Lunel, *Numerical approximation of delay differential equations with small solutions*, Proceedings of 16th IMACS World Congress on Scientific Computation, Applied Mathematics and Simulation, Lausanne 2000, paper 173-3, New Brunswick, 2000 (ISBN 3-9522075-1-9).

[11] N. J. Ford, P. M. Lumb, P. M. Lima, M. F. Teodoro, *The numerical solution of forward-backward equations: decomposition and related issues*, Journal of Computational and Applied Mathematics, **234** (2010), 9, 2826-2837.

[12] D. Henry, *Small solutions of linear autonomous functional differential equations*, Journal of Differential Equations **8** (1970), 494-501.

[13] V. Iakovleva and C. Vanegas , *On the solution of differential equations with delayed and advanced arguments*, Electronic Journal of Differential Equations, Conference 13 (2005),57-63.

[14] P. M. Lima, M. F. Teodoro, N.J. Ford and P.M. Lumb, *Analytical and Numerical Investigation of Mixed Type Functional Differential Equations*, Journal of Computational and Applied Mathematics, **234** (2010), 9, 2732-2744.

[15] P. M. Lima, M. F. Teodoro, N. J. Ford and P. M. Lumb, *Finite Element Solution of a Linear Mixed-Type Functional Differential Equation*, Numerical Algorithms, **55** (2010), 301-320.

[16] P. M. Lima, M. F. Teodoro, N. J. Ford and P. M. Lumb, *Analysis and Computational Approximation of a Forward-Backward Equation Arising in Nerve Conduction*, accepted for publication in Proceedings of the International Conference on Differential & Difference Equations, Azores, 2011.

[17] P. M. Lumb, *Delay differential equations: Detection of small solutions*, PhD Thesis, University of Liverpool, UK, 2004.

[18] J. Mallet-Paret, *The Fredholm alternative for functional differential equations of mixed type*, Journal of Dynamics and Differential Equations, **11** (1999), 1, 1-47.

[19] J. Mallet-Paret, *The Global Structure of Traveling Waves in Spattially Discrete Dynamical Systems*, Journal of Dynamics and Differential Equations, **11** (1999), 1, 49-128.

[20] L. S. Pontryagin, R. V.gamkreledze, E. F. Mischenko, *The mathematical Theory of Optimal Process*, Interscience, New York (1962).

[21] A. Rustichini, *Functional differential equations of mixed type: The linear autonomous case*, Journal of Dynamics and Differential Equations **1** (1989), 2, 121-143.

[22] A. Rustichini, *Hopf bifurcation for functional differential equations of mixed type*, Journal of Dynamics and Differential Equations **1** (1989), 2, 145-177.

[23] C. da Silva and R. Escalante, *Segmented Tau approximation for a forward-backward functional differential equation*, Computers and Mathematics with Applications **62** (2011), 4582-4591.

[24] M. F. Teodoro, N. J. Ford, P. M. Lima, and P. M. Lumb, *New approach to the numerical solution of forward-backward equations*, Front. Math. China, **4** (2009), 1, 155-168.

[25] M. F. Teodoro, P. M. Lima, N. J. Ford and P. M. Lumb, *Numerical modelling of a functional differential equation with deviating arguments using a collocation method*, Proceedings of ICNAAM 2008, International Conference on Numerical Analysis and Applied Mathematics, Kos 2008, AIP Proceedings, **1048** (2008), 553-557.

[26] M. F. Teodoro, P. M. Lima, N. J. Ford and P.M. Lumb, *Numerical Approximation of Forward-Backward Differential Equations by a Finite Element Method*, Proceedings of CMMSE 2009, 9th International Computational and Mathematical Methods in Science and Engineering, Gijon, Spain, **3** (2009), 1010-1019, ISBN:978-84-612-9727-6.

[27] J.L. Pons, *Wearable Robots: Biomechatronic Exoskeletons*, Wiley Blackwell, 2008.

[28] J. E. Sanders, B. S. Goldstein & D. F. Leotta, *Skin response to mechanical stress: adaptation rather than breakdown - a literature review*, Journal of Rehabilitation Research and Development **32** 214–226 (1995).

[29] G. A. F. Seber, C. J. Wild, *Nonlinear Regression* , Wiley Blackwell, 1989.

[30] P. Silva, *Computational Modelling Of A Wearable Ankle-Foot Orthosis For Locomotion Analysis And Comfort Evaluation*, PHD Thesis, Department of Mechanical Engineering, Instituto Superior Técnico, Universidade Técnica de Lisboa, 2012.

[31] P. Silva, A. Monteiro, I. Bernardo, R. Claudio & C. Figueiredo-Pina, *Measuring Discomfort: From Pressure Pain Threshold To Soft Tissues Deformation*, Journal of Biomechanics **45** S1 S576 (2012).